XXXVI. *A Diſquiſition concerning certain Fluents, which are aſſignable by the Arcs of the Conic Sections; wherein are inveſtigatea ſome* new *and* uſeful *Theorems for computing ſuch Fluents: By* John Landen, *F. R. S.*

Read June 6, 1771.

MR. Mac Laurin, in his Treatiſe of Fluxions, has given ſundry very elegant Theorems for computing the Fluents of certain Fluxions by means of Elliptic and Hyperbolic Arcs; and Mr. D'Alembert, in the Memoirs of the Berlin Academy, has made ſome improvement upon what had been before written on that ſubject, But ſome of the Theorems given by thoſe Gentlemen being in part expreſſed by the difference between an Arc of an Hyperbola and its Tangent, and ſuch difference being not directly attainable, when ſuch Arc and its Tangent both become infinite, as they will do when the *whole* Fluent is wanted, although ſuch Fluent be at the ſame time finite; thoſe Theorems therefore in that caſe fail, a computation thereby being then impracticable, without ſome farther help.

The ſupplying that defect I conſidered as a point of ſome importance in Geometry, and therefore I earneſtly

Fig. 1.

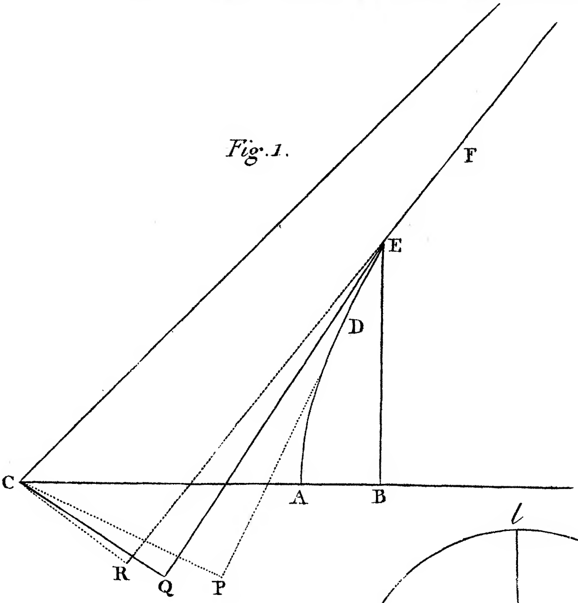


Fig. 2.

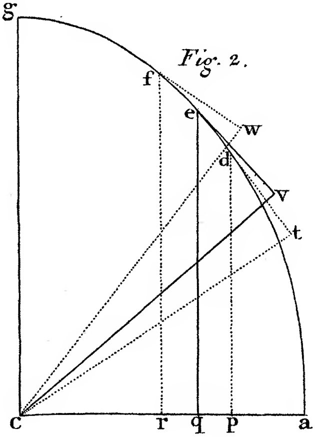
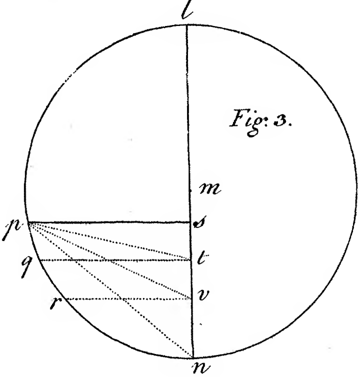


Fig. 3.



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earneſtly wiſhed, and endeavoured, to accompliſh that buſineſs; my aim being to aſcertain, by means of ſuch arcs, as above-mentioned, the *Limit* of the difference between the Hyperbolic Arc and its Tangent, whilſt the point of contact is ſuppoſed to be carried to an infinite diſtance from the vertex of the curve, ſeeing that, by the help of that *Limit*, the computation would be rendered practicable in the caſe wherein, without ſuch help, the before-mentioned Theorems fail. And having ſucceeded to my ſatisfaction, I preſume, the reſult of my endeavours, which this Paper contains, will not be unacceptable to the Royal Society.

1.

Suppoſe the curve ADEF (Tab. XII. fig. 1.) to be a conic *Hyperbola*, whoſe ſemi-tranſverſe axis AC is $= m$, and ſemi-conjugate $= n$.

Let CP, perpendicular to the tangent DP, be called p; and put $f = \frac{m^2 - n^2}{2m}$, $z = \frac{p^2}{m}$. Then (as is well known) will DP — AD be = the fluent of $\frac{-\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$, p and z being each = to m when AD is $= 0$.

2.

Suppoſe the curve adefg (fig. 2.) to be a quadrant of an *Ellipſis*, whoſe ſemi-tranſverſe axis cg is $= \sqrt{m^2 + n^2}$, and ſemi-conjugate ac $= n$. Let ct be perpendicular to the tangent dt, and let the abſciſſa cp be $= n \times \overline{\frac{z}{m}}\Big|^{\frac{1}{2}}$. Then will the ſaid tangent dt be $= m \times \overline{\frac{mz - z^2}{n^2 + mz}}\Big|^{\frac{1}{2}}$; and the fluxion thereof will be found

$$= \frac{1}{2} mn^2 z^{-\frac{1}{2}} \dot{z} \times \frac{\overline{m - z}|^{\frac{1}{2}}}{\overline{n^2 + mz}|^{\frac{3}{2}}} - \frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}.$$

3.

In the expreſſion $\frac{y^{q-1} \dot{y}}{\overline{a + by}|^{r} \times \overline{c + dy}|^{s}}$, let $\frac{c + dy}{a + by}$ be ſuppoſed $= z$. Then will $\frac{az - c}{d - bz}$ be $= y$, and the propoſed expreſſion will be

$$= \frac{\overline{ad - bc}|^{1 - r - s} \times z^{-s} \dot{z}}{\overline{az - c}|^{1 - q} \times \overline{d - bz}|^{1 + q - r - s}}.$$

4.

Taking, in the laſt article, r and s each $= \frac{1}{2}$, $q = \frac{3}{2}$, $a = -d = \frac{n^2}{m}$, $b = 1$, and $c = n^2$, we have

$$\frac{y^{\frac{1}{2}} \dot{y}}{\overline{\frac{n^2}{m} + y}\Big|^{\frac{1}{2}} \overline{\times n^2 - \frac{n^2}{m} y}\Big|^{\frac{1}{2}}} \left(= \frac{m^{\frac{1}{2}} n^{-1} y^{\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}} \right)$$

$$= - mnz^{-\frac{1}{2}} \dot{z} \times \frac{\overline{m - z}|^{\frac{1}{2}}}{\overline{n^2 + mz}|^{\frac{3}{2}}}.$$

It

It appears therefore, that, y being $= n^2 \times \frac{m-z}{n^2+mz}$,

$$-\frac{\frac{1}{2}m^{\frac{1}{2}}y^{\frac{1}{2}}\dot{y}}{\sqrt{n^2+2fy-y^2}} - \frac{\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2+2fz-z^2}} \text{ is}$$

$$= \frac{1}{2}mn^2z^{-\frac{1}{2}}\dot{z} \times \frac{\overline{m-z}|^{\frac{1}{2}}}{\overline{n^2+mz}|^{\frac{3}{2}}} - \frac{\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2+2fz-z^2}};$$

which, by Art. 2. is = the *fluxion of the tang.* dt.

Conſequently, taking the fluents, by Art. 1. and correcting them properly, we find
DP — AD + FR — AF = L + dt.

CP, in fig. 1. being $=m^{\frac{1}{2}}z^{\frac{1}{2}}$; cp, in fig. 2. $=n \times \overline{\frac{z}{m}}|^{\frac{1}{2}}$;

CR, perpendicular to the tangent FR $= m^{\frac{1}{2}}y^{\frac{1}{2}}$;

DP — AD = the *fluent* of $\frac{-\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2+2fz-z^2}}$;

FR — AF = the fluent of $\frac{-\frac{1}{2}m^{\frac{1}{2}}y^{\frac{1}{2}}\dot{y}}{\sqrt{n^2+2fy-y^2}}$;

and L the *Limit* to which the difference DP — AD, or FR — AF, approaches, upon carrying the point D, or F, from the vertex A *ad infinitum.*

5.

Suppoſe y equal to z, and that the points D and F then coincide in E, the points d and p being at the ſame time in e and q reſpectively. Then cv being perpendicular to the tangent ev, that tangent will be a *maximum* and equal to cg — ac $= \sqrt{m^2+n^2} - n$; the tangent EQ (in the hyperbola) will be $= \sqrt{m^2+n^2}$;

the

the abſciſſa BC $= m\sqrt{1+\frac{n}{\sqrt{m^2+n^2}}}$; the ordinate BE $= n \times \sqrt{\frac{n}{\sqrt{m^2+n^2}}}$; and it appears, that L is $= 2\text{EQ} - 2\text{AE} - \text{ev} = n + \sqrt{m^2+n^2} - 2\text{AE}$! Thus the *Limit* which I propoſed to aſcertain is inveſtigated, m and n being any right lines whatever!

6.

The *whole fluent* of $\frac{\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2+2fz-z^2}}$, generated whilſt z from o becomes $= m$, being equal to L; and the fluent of the ſame fluxion (ſuppoſing it to begin when z begins) being in general equal to $L + \text{AD} - \text{DP} = \text{FR} - \text{AF} - \text{dt}$; it appears, that, k being the value of z correſponding to the fluent $L + \text{AD} - \text{DP}$, $\frac{mn^2 - n^2k}{n^2+mk}$ will be the value of z correſponding to the fluent $L + \text{AF} - \text{FR}$, and $\text{FR} - \text{AF}$ will be the part generated whilſt z from $\frac{mn^2 - n^2k}{n^2+mk}$ becomes $= m$. It follows therefore, that the *tang*. dt, together with the fluent of $\frac{\frac{3}{4}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2+2fz-z^2}}$ generated whilſt z from o becomes equal to any quantity k, is equal to the fluent of the ſame fluxion generated whilſt z from $\frac{mn^2 - n^2k}{n^2+mk}$ becomes $= m$; cp being taken $= n \times \left.\frac{k}{m}\right|^{\frac{1}{2}}$.

Suppoſe

Suppoſe $k = \frac{mn^2 - n^2 k}{n^2 + mk}$, its value will then be $\frac{n}{m}\sqrt{m^2 + n^2} - \frac{n^2}{m}$. Conſequently the fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ generated whilſt z from o becomes $= \frac{n}{m}\sqrt{m^2 + n^2} - \frac{n^2}{m}$, together with the quantity $\sqrt{m^2 + n^2} - n$, is equal to the fluent of the ſame fluxion generated whilſt z from $\frac{n}{m}\sqrt{m^2 + n^2} - \frac{n^2}{m}$ becomes $= m$: and theſe two parts of the *whole fluent* being denoted by M and N reſpectively; M will be $= n -$ AE, and N $= \sqrt{m^2 + n^2} -$ AE.

7.

The fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ being $L +$ AD $-$ DP, the fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}} +$ DP $-$ AD $- L$ will be $= o$.

Therefore, the fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}} +$ the fluent of $\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ being $=$ the fluent of $\frac{1}{2} z^{-\frac{1}{2}} \dot{z} \times \left.\overline{\frac{n^2 + mz}{m - z}}\right|^{\frac{1}{2}}$, it is obvious, that the fluent of $\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ is $=$ DP $-$ AD $- L +$ the fluent of $\frac{1}{2} z^{-\frac{1}{2}} \dot{z} \times \left.\overline{\frac{n^2 + mz}{m - z}}\right|^{\frac{1}{2}}$

$=$ DP

$= \mathrm{DP} - \mathrm{AD} - L +$ the *elliptic arc* dg (fig. 2.) whoſe abſciſſa cp is $= n \times \overline{\frac{z}{m}}\Big|^{\frac{1}{2}}$.

Conſequently, putting E for $\frac{1}{4}$ of the periphery of that ellipſis, it appears that the *whole fluent* of $\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$, generated whilſt z from o becomes $= m$, is equal to $E - L = E + 2\mathrm{AE} - n - \sqrt{m^2 + n^2}$.

8.

By taking, in Art. 3. q, r, and s, each $= \frac{1}{2}$; and $a = -d = \frac{n^2}{m}$, $b = 1$, and $c = n^2$; we find, that, if y be $= \frac{mn^2 - n^2 z}{n^2 + mz}$, $\frac{z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}} + \frac{y^{-\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}}$ will be $= o$.

It is obvious therefore, that the fluent of $\frac{z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$, generated whilſt z from o becomes equal to any quantity k, is equal to the fluent of the ſame fluxion, generated whilſt z from $\frac{mn^2 - n^2 k}{n^2 + mk}$ becomes $= m$.

Now, ſuppoſing $k = \frac{mn^2 - n^2 k}{n^2 + mk}$, its value will be $\frac{n}{m}\sqrt{m^2 + n^2} - \frac{n^2}{m}$.

Conſequently, the fluent of $\frac{z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$, generated whilſt z from o becomes $= \frac{n}{m}\sqrt{m^2 + n^2} - \frac{n^2}{m}$, is equal to *half* the fluent of the ſame fluxion, generated whilſt z from o becomes $= m$; which *half fluent* is known by the preceding article.

9.

It appears, by Ar. 4. that

$$\frac{\frac{1}{2} m^{\frac{1}{2}} y^{\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}} + \frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}} \text{ is} = -\text{the } \textit{flux. of the tang.}\ \mathrm{dt};$$

and it appears, by the laſt article, that

$$\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 y^{-\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}} + \frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}} \text{ is} = o;$$

$mn^2 - n^2y - n^2z - myz$ being $= o$.

Therefore, by addition, we have

$$\tfrac{1}{2} y^{-\frac{1}{2}} \dot{y} \times \overline{\frac{n^2 + my}{m - y}}\Big|^{\frac{1}{2}} + \tfrac{1}{2} z^{-\frac{1}{2}} \dot{z} \times \overline{\frac{n^2 + mz}{m - z}}\Big|^{\frac{1}{2}}$$

$=$ — the *fluxion of the tangent* dt.

Conſequently, by taking the correct fluents, we find the *tang.* dt ($=$ the *tang.* fw) $=$ the *arc* ad — the *arc* fg! the abſciſſa cp being $= n \times \overline{\frac{z}{m}}\Big|^{\frac{1}{2}}$, the abſciſſa cr $= n \times \overline{\frac{y}{m}}\Big|^{\frac{1}{2}}$, and their relation expreſſed by the equation $n^6 - n^4u^2 - n^4v^2 - m^2u^2v^2 = o$, u and v being put for cp and cr reſpectively. Moreover, the tangents dt, fw, will each be $= \frac{m^2uv}{n^4}$; and ct $\times$ cw $= \overline{\mathrm{cv}}^2 =$ ac $\times$ cg.

If for the ſemi-tranſverſe axis cg we ſubſtitute h nſtead of $\sqrt{m^2 + n^2}$, the relation of u to v will be expreſſed by the equation

$$n^6 - n^4u^2 - n^4v^2 - \overline{h^2 - n^2} \times u^2v^2 = o,$$ and

dt (= fw) will be $= \frac{h^2 - n^2}{n^3} \times uv$.

If u and v be reſpectively put for fr and dp, their relation will be expreſſed by the equation

$$h^6 - h^4u^2 - h^4v^2 + \overline{h^2 - n^2} \times u^2v^2 = o,$$ and

dt (= fw) will be $= \frac{h^2 - n^2}{h^3} \times uv$.

10.

Suppoſe y = to z, (that is, $v = u$) and that the points d and f coincide in e. In which caſe the tangent dt will be a *maximum*, and = cg — ac. It appears then that the *arc* ae — the *arc* eg is = cg — ac.

Conſequently, putting E for the quadrantal arc ag, we find that the *arc* ae is $= \frac{E + h - n}{2}$!

the *arc* eg $= \frac{E - h + n}{2}$!

There are, I am aware, ſome other parts of the *arc* ag, whoſe lengths may be aſſigned by means of the whole length (ag) with right lines; but to inveſtigate ſuch other parts is not to my preſent purpoſe.

11.

Taking m and n each = 1; that is, ac (= AC) = 1, and cg = $\sqrt{2}$; let the arc ag be then expreſſed by e: put c for $\frac{1}{4}$ of the periphery of the circle whoſe radius is 1; and let the *whole fluents* of $\frac{\frac{1}{2} z^{\frac{1}{2}} \dot{z}}{\sqrt{1 - z^2}}$ and $\frac{\frac{1}{2} z^{-\frac{1}{2}} \dot{z}}{\sqrt{1 - z^2}}$, generated whilſt z from o becomes $= 1$, be denoted by F and G reſpectively. Then, by what is ſaid above, $F + G$ will be $= e$; and, by my theorem for comparing curvilineal areas, or fluents, publiſhed in the *Philoſ. Tranſact.* for the year 1768, it appears that $F \times G$ is $= \frac{1}{2} c$. From which equations we find $F = \frac{1}{2} e - \frac{1}{2} \sqrt{e^2 - 2c}$, and $G = \frac{1}{2} e + \frac{1}{2} \sqrt{e^2 - 2c}$.

But m and n being each $= 1$, L is $= F$; therefore $1 + \sqrt{2} - 2AE$, the value of L, from Art. 5. is, in this caſe, $= \frac{1}{2} e - \frac{1}{2} \sqrt{e^2 - 2c}$. Conſequently, in the *equilateral hyperbola*, the *arc* AE, whoſe abſciſſa BC is $= \sqrt{1 + \frac{1}{\sqrt{2}}}$, will be $= \frac{1}{2} + \frac{1}{\sqrt{2}} - \frac{1}{4} e + \frac{1}{4} \sqrt{e^2 - 2c}$, by what is ſaid in the article laſt mentioned. Hence the *rectification* of *that arc* may be effected by means of the *circle* and *ellipſis*!

The application of theſe *Improvements* will be eaſily made by the intelligent Reader, who is acquainted with what has been before written on the ſubject. But there is a theorem (demonſtrable by what is proved in Art. 8.) ſo remarkable, that I cannot conclude this diſquiſition without taking notice of it.

12.

Let *lpqn* (fig. 3.) be a circle perpendicular to the horizon. whoſe higheſt point is *l*, loweſt *n*, and center *m*: let *p* and *q* be any points in the ſemi-circumference *lpqn*: draw *ps*, *qt* parallel to the horizon, interſecting *lmn* in *s* and *t*; and, having joined *lp*, *pt*, make the angle *lpv* equal to *ltp*, and draw *rv* parallel to *qt*, interſecting the circle in *r*, and the diameter *lmn* in *v*. Let a pendulum, or other heavy body, deſcend by its gravity from *p* along the arc *pqrn*: the body ſo deſcending will paſs over the arc *pq* exactly in the ſame time as it will paſs over the arc *rn*; and therefore, *qt* and *rv* coinciding when *lt* is equal to *lp*, it is evident that the time of deſcent from *p* to *q* will then be preciſely equal to *half* the time of deſcent from *p* to *n*!

And it is farther obſervable, that, if *pqn* be a quadrant, the *whole* time of deſcent will be $=\overline{\tfrac{a}{b}}\Big)^{\frac{1}{2}} \times \overline{\tfrac{1}{2}e + \tfrac{1}{2}\sqrt{e^2 - 2c}}$; the radius *lm*, or *mn*, being $=a$; and *b* being put (for $16\frac{1}{12}$ feet) the ſpace a heavy body deſcending freely from reſt falls through in one ſecond of time.

In general, *ns* being denoted by *d*, and the diſtance of the body from the line *ps*, in its deſcent, by *x*, the fluxion of the time of deſcent will be expreſſed by $\dfrac{\frac{1}{2}ab^{-\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}}{\sqrt{2ad - d^2 - \overline{2a - 2d}\,.\,x - x^2}}$; the fluent whereof, correſponding to any value of *x*, may be obtained by Art. 7. By which article it appears, that the *whole* time

time of deſcent from any point p will be

$$= \frac{a}{b^{\frac{1}{2}} d^{\frac{1}{2}} \times \overline{2a - d}} \times \overline{E + 2\mathrm{AE} - pn - ps}.$$

The ſemi-tranſverſe A C (fig. 1.) being $= ns$;
the ſemi-tranſverſe c g (fig. 2.) $= np$;
and the ſemi-conjugate in each figure $= ps$.

Since writing the above, I have diſcovered a general theorem for the rectification of the Hyperbola, by means of two Ellipſes; the inveſtigation whereof I purpoſe to make the ſubject of another Paper.

XXXVII. *A*